According to the Latest Syllabus

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Mathematics

Class-11 (Part-II)

For the Students of Rajasthan Board of Secondary Education

By

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Preface

We are extremely pleased to present this book according to latest syllabus of NCERT. The book has been written in easy and simple language so that students may assimilate the subject easily. We hope that students will get benefitted from it and teachers will appreciate our efforts. In comparison to other books available in market, this book has many such features which make it a unique book :

1. Theoretical subject-material is given in adequate and accurate description along with pictures.
2. The latest syllabus of NCERT is followed thoroughly.
3. Complete solutions of all the questions given at the end of the chapter in the textbook are given in easy language.
4. Topic wise summary is also given in each chapter for the revision of the chapter.
5. In every chapter, all types of questions that can be asked in the exam (Objective, Fill in the blanks, Very short, Short, Numerical and Long answer type questions) are given.
6. At the end of every chapter, multiple choice questions asked in various competitive exams are also given with solutions.

Valuable suggestions received from subject experts, teachers and students have also been given appropriate place in the book.

We wholeheartedly bow to the Almighty God, whose continuous inspiration and blessings have made the writing of this book possible.

We express our heartfelt gratitude to the publisher – Mr. Pradeep Mittal and Manoj Mittal of Sanjiv Prakashan, all their staff, laser type center and printer for publishing this book in an attractive format on time and making it reach the hands of the students.

Although utmost care has been taken in publishing the book, human errors are still possible, hence, valuable suggestions are always welcome to make the book more useful.

In anticipation of cooperation!

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Contents

CHAPTER 8

SEQUENCES AND SERIES

Chapter Overview

8.1 Introduction

Generally in the English language, the word 'sequence' means a collection of objects in which each object is ordered in such a way that it has an identified first member, second member, third member and so on. In mathematics, the word sequence is used in the same sense as that of in the English language.

By the word 'Series', we mean those sequences whose elements follow a particular pattern or rule. In the pervious class, we have studied arithmetic progression. In this chapter along with arithmetic progression we will study about arithmetic mean, geometric mean, the relationship between arithmetic mean and geometric mean and in special series sum of natural numbers, sum of squares of natural numbers and also about the sum of cubes of natural numbers.

8.2 Sequences

The arrangement of numbers which is done in a certain order according to some rule *i.e.* if the quantities are in particular order as per certain rules, then it is called sequence. Each element of the sequence is called term.

Some Examples of Sequences :

(i) Consider 1, 3, 5, 7, 9, ... in this sequence each term is obtained by adding '2' to the previous term. The n^{th} term of this sequence can be written as $a_n = (2n - 1)$, where n is a natural number. In this way, in short $a_1 + a_2 + a_3 + ... a_n = \sum_{k=1}^{n} a_k$.

(ii) Consider a sequence 3, –9, 27, –81............. . In this sequence each term is obtained by multiplying the previous term by '–3'. The n^{th} term of the sequence can be written as $a_n = (-1)^{n-1}. 3^n$, where n is a natural number.

8.3 Types of Sequences

There are two types of sequences :
(i) Finite sequence
(ii) Infinite sequence

If the number of terms in a sequence is fixed then it is called a finite sequence. If the number of terms in a sequence is indefinite, then it is called an infinite sequence.

8.4 Series

If a_n is a sequence, then the expression $a_1 + a_2 + a_3 + ...$, is called a series.

In other words it represents the sum of terms a_1, a_2, a_3, ... a_n.

8.5 Types of Series

There are two types of series : (i) Finite Series and (ii) Infinite Series.

The series in which number of terms are limited is called the finite series and the series in which the number of terms is infinite is called infinite series.

A series can be represented in a compact form, called summation or sigma notation. The Greek capital letter, $\sum$ (sigma), is used to represent the sum. So, series $a_1 + a_2 + a_3 + ... + a_n$.

Example 1. Write the first three terms in each of the following sequences defined by the following : (NCERT)

(i) $a_n = 2n + 5$ (ii) $a_n = \frac{n-3}{4}$

Sol. (i) Here,

$$a_n = 2n + 5$$

Putting $n = 1, 2$ and 3

$$a_1 = 2 \times 1 + 5 = 2 + 5 = 7$$
$$a_2 = 2 \times 2 + 5 = 4 + 5 = 9$$
$$a_3 = 2 \times 3 + 5 = 6 + 5 = 11$$

Therefore the required terms are 7, 9 and 11.

(ii) Here,

$$a_n = \frac{n-3}{4}$$

Putting $n = 1, 2$ and 3

$$a_1 = \frac{1-3}{4} = \frac{-2}{4} = \frac{-1}{2}$$

$$a_2 = \frac{2-3}{4} = \frac{-1}{4}$$

$$a_3 = \frac{3-3}{4} = \frac{0}{4} = 0$$

So, first three terms are $\frac{-1}{2}$, $\frac{-1}{4}$ and 0.

Example 2. What is the 20th term of the sequence defined by $a_n = (n - 1)(2 - n)(3 + n)$? (NCERT)

Sol. Here,

$$a_n = (n - 1)(2 - n)(3 + n)$$

Putting $n = 20$

$$a_{20} = (20 - 1)(2 - 20)(3 + 20)$$
$$= (19) \times (-18) \times (23)$$
$$= -7866$$

Example 3. Let the sequence a_n be defined as follows :

$$a_n = 1$$
$$a_n = a_{n-1} + 2 \text{ for } n \geq 2$$

Find first five terms and write corresponding series. (NCERT)

Sol. Here, $a_1 = 1$

$a_n = a_{n-1} + 2$ for $n \geq 2$

Here value of n is 3 and greater than 2.

So, $a_2 = a_{2-1} + 2 = a_1 + 2$

Putting value of a_1

Then $a_2 = 1 + 2 = 3$

Now $a_3 = a_{3-1} + 2 = a_2 + 2 = 3 + 2 = 5$

$a_4 = a_{4-1} + 2 = a_3 + 2 = 5 + 2 = 7$

$a_5 = a_{5-1} + 2 = a_4 + 2 = 7 + 2 = 9$

Hence, the first five terms of the sequence are 1, 3, 5, 7 and 9. The corresponding series is $1 + 3 + 5 + 7 + 9 + ...$.

Example 4. Find the first four terms of the sequences whose n^{th} terms are given by :

(i) $\frac{(n+1)^2}{n}$ (ii) $2n^2 - n + 2$

Sol. (i) Given, $a_n = \frac{(n+1)^2}{n}$

Putting $n = 1, 2, 3, 4$

$$a_1 = \frac{(1+1)^2}{1} = \frac{(2)^2}{1} = \frac{4}{1} = 4$$

$$a_2 = \frac{(2+1)^2}{2} = \frac{(3)^2}{2} = \frac{9}{2} = \frac{9}{2}$$

$$a_3 = \frac{(3+1)^2}{3} = \frac{(4)^2}{3} = \frac{16}{3}$$

$$a_4 = \frac{(4+1)^2}{4} = \frac{5^2}{4} = \frac{25}{4}$$

So first four terms of the series are 4, $\frac{9}{2}$, $\frac{16}{3}$ and $\frac{25}{4}$.

(ii) Given, $a_n = 2n^2 - n + 2$

Putting $n = 1, 2, 3, 4$

$$a_1 = 2(1)^2 - 1 + 2 = 2 - 1 + 2 = 3$$

$a_2 = 2(2)^2 - 2 + 2 = 2 \times 4 = 8$
$a_3 = 2(3)^2 - 3 + 2 = 18 - 3 + 2 = 17$
$a_4 = 2(4)^2 - 4 + 2 = 32 - 4 + 2 = 30$

So first four terms of the given series are 3, 8, 17 and 30.

Example 5. If $T_n = an^2 + bn + c$ and $T_1 = 10$, $T_2 = 19$ and $T_3 = 32$, then find the values of a, b and c.

Sol. Given, $T_1 = 10$, $T_2 = 19$ and $T_3 = 32$

and $T_n = an^2 + bn + c$

$\therefore$ $T_1 = a(1)^2 + b(1) + c = a + b + c$

$\therefore$ $a + b + c = 10$ ....(i)

$\because$ $T_1 = 10$

$T_2 = 19 = a(2)^2 + b(2) + c$

$19 = 4a + 2b + c$ ....(ii)

$T_3 = 32 = a(3)^2 + b(3) + c$

$32 = 9a + 3b + c$ ....(iii)

Subtracting equation (i) from (ii)

$3a + b = 9$ ....(iv)

Subtracting equation (ii) from (iii)

$5a + b = 13$ ....(v)

Subtracting equation (iv) from (v)

$2a = 4 \quad \Rightarrow a = 2$

From equation (iv) :

$3 \times 2 + b = 9 \;\Rightarrow b = 3$

From equation (i) :

$a + b + c = 10$

$2 + 3 + c = 10 \;\Rightarrow c = 5$

Hence, $a = 2$, $b = 3$ and $c = 5$ **Ans.**

Exercise 8.1

Write the first five terms of each of the sequences in Exercises 1 to 6 whose n^{th} terms are :

1. $a_n = n\,(n + 2)$

Sol. $\because$ $a_n = n\,(n + 2)$

Putting $n = 1$, First term $a_1 = 1\,(1 + 2) = 3$
Putting $n = 2$, Second term $a_2 = 2\,(2 + 2) = 8$
Putting $n = 3$, Third term $a_3 = 3\,(3 + 2) = 15$
Putting $n = 4$, Fourth term $a_4 = 4\,(4 + 2) = 24$
Putting $n = 5$, Fifth term $a_5 = 5\,(5 + 2) = 35$
$\therefore$ First five terms are 3, 8, 15, 24 and 35. **Ans.**

2. $a_n = \dfrac{n}{n+1}$

Sol. $\because$ $a_n = \dfrac{n}{n+1}$

Putting $n = 1$, First term $a_1 = \dfrac{1}{1+1} = \dfrac{1}{2}$

Putting $n = 2$, Second term $a_2 = \dfrac{2}{2+1} = \dfrac{2}{3}$

Putting $n = 3$, Third term $a_3 = \dfrac{3}{3+1} = \dfrac{3}{4}$

Putting $n = 4$, Fourth term $a_4 = \dfrac{4}{4+1} = \dfrac{4}{5}$

and Putting $n = 5$, Fifth term $a_5 = \dfrac{5}{5+1} = \dfrac{5}{6}$

Hence, first five terms are $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}$ and $\dfrac{5}{6}$. **Ans.**

3. $a_n = 2^n$

Sol. $\because$ $a_n = 2^n$

Putting $n = 1$, First term $a_1 = 2^1 = 2$
Putting $n = 2$, Second term $a_2 = 2^2 = 4$
Putting $n = 3$, Third term $a_3 = 2^3 = 8$
Putting $n = 4$, Fourth term $a_4 = 2^4 = 16$
Putting $n = 5$, Fifth term $a_5 = 2^5 = 32$
Hence, first five terms are 2, 4, 8, 16 and 32. **Ans.**

4. $a_n = \dfrac{2n-3}{6}$

Sol. $\because$ $a_n = \dfrac{2n-3}{6}$

Putting $n = 1$, First term $a_1 = \dfrac{2(1)-3}{6} = \dfrac{2-3}{6} = -\dfrac{1}{6}$

Putting $n = 2$, Second term $a_2 = \dfrac{2(2)-3}{6} = \dfrac{4-3}{6} = \dfrac{1}{6}$

Putting $n = 3$ Third term $a_3 = \dfrac{2(3)-3}{6} = \dfrac{6-3}{6} = \dfrac{3}{6} = \dfrac{1}{2}$

Putting $n = 4$, Fourth term $a_4 = \dfrac{2(4)-3}{6} = \dfrac{8-3}{6} = \dfrac{5}{6}$

Putting $n = 5$, Fifth term $a_5 = \dfrac{2(5)-3}{6} = \dfrac{10-3}{6} = \dfrac{7}{6}$

Hence, first five terms are $-\dfrac{1}{6}, \dfrac{1}{6}, \dfrac{1}{2}, \dfrac{5}{6}$ and $\dfrac{7}{6}$. **Ans.**

5. $a_n = (-1)^{n-1}\, 5^{n+1}$

Sol. $\because$ $a_n = (-1)^{n-1}\, 5^{n+1}$

Putting $n = 1$, $a_1 = (-1)^{1-1} \,.\, 5^{1+1} = 25$
Putting $n = 2$, $a_2 = (-1)^{2-1} \,.\, 5^{2+1} = -125$
Putting $n = 3$, $a_3 = (-1)^{3-1} \,.\, 5^{3+1} = 625$
Putting $n = 4$, $a_4 = (-1)^{4-1} \,.\, 5^{4+1} = -3125$
Putting $n = 5$ $a_5 = (-1)^{5-1} \,.\, 5^{5+1} = 15625$
Hence, first five terms are 25, –125, 625, –3125 and 15625. **Ans.**

6. $a_n = n\dfrac{(n^2+5)}{4}$

Sol. Given, $a_n = n\dfrac{(n^2+5)}{4}$

Putting $n = 1$, $a_1 = 1.\dfrac{(1^2+5)}{4} = \dfrac{1\times 6}{4} = \dfrac{3}{2}$

Putting $n = 2$, $a_2 = 2.\dfrac{(2^2+5)}{4} = \dfrac{2\times 9}{4} = \dfrac{9}{2}$

Putting $n = 3$, $a_3 = 3.\dfrac{(3^2+5)}{4} = \dfrac{3\times 14}{4} = \dfrac{21}{2}$

Putting $n = 4$, $a_4 = 4.\dfrac{(4^2+5)}{4} = \dfrac{4\times 21}{4} = 21$

Putting $n = 5$, $a_5 = 5.\dfrac{(5^2+5)}{4} = \dfrac{5\times 30}{4} = \dfrac{75}{2}$

Hence, first five terms are $\frac{3}{2}, \frac{9}{2}, \frac{21}{2}$, 21 and $\frac{75}{2}$. **Ans.**

Find the indicated terms in each the sequences in exercises 7 to 10 where n^{th} terms are :

7. $a_n = 4n - 3; a_{17}, a_{24}$

Sol. $\because a_n = 4n - 3$

Putting $n = 17$, $a_{17} = 4 . 17 - 3 = 68 - 3 = 65$

Putting $n = 24$, $a_{24} = 4 . 24 - 3 = 96 - 3 = 93$

$\therefore \quad a_{17} = 65$ and $a_{24} = 93$ **Ans.**

8. $a_n = \frac{n^2}{2^n}; a_7$

Sol. $\because a_n = \frac{n^2}{2^n}; a_7$

For a_7, putting $n = 7$, $a_7 = \frac{7^2}{2^7} = \frac{49}{128}$

So, $a_7 = \frac{49}{128}$ **Ans.**

9. $a_n = (-1)^{n-1} n^3; a_9$

Sol. $\because a_n = (-1)^{n-1} n^3$

For a_9, putting $n = 9, a_9 = (-1)^{9-1} . (9)^3$

$a_9 = (-1)^8 . 9 \times 9 \times 9 = 729$

So, $a_9 = 729$ **Ans.**

10. $a_n = \frac{n(n-2)}{n+3}; a_{20}$

Sol. $\because a_n = \frac{n(n-2)}{n+3}; a_{20}$

For a_{20}, putting $n = 20$, $a_{20} = \frac{20(20-2)}{20+3}$

$= \frac{20 \times 18}{23} = \frac{360}{23}$

So, $a_{20} = \frac{360}{23}$ **Ans.**

Write the first five terms of each of the sequences in Exercises 11 to 13 and obtain the corresponding series :

11. $a_1 = 3, a_n = 3a_{n-1} + 2$ **for all** $n > 1$

Sol. $\because \quad a_1 = 3,$

$a_n = 3a_{n-1} + 2 \ \forall \ n > 1$

For $n = 2$ $\quad a_2 = 3a_{2-1} + 2 = 3a_1 + 2$

$= 3.3 + 2 = 9 + 2 = 11$

For $n = 3$ $\quad a_3 = 3a_{3-1} + 2 = 3a_2 + 2$

$= 3.11 + 2 = 33 + 2 = 35$

For $n = 4$ $\quad a_4 = 3a_{4-1} + 2 = 3a_3 + 2$

$= 3.35 + 2 = 105 + 2 = 107$

For $n = 5$ $\quad a_5 = 3a_{5-1} + 2 = 3a_4 + 2$

$= 3.107 + 2 = 321 + 2 = 323$

So, first five terms are = 3, 11, 35, 107, 323 and sequence = 3 + 11 + 35 + 107 + 323 + ... **Ans.**

12. $a_1 = -1, a_n = \frac{a_{n-1}}{n}$, **where** $n \geq 2$

Sol. $\because \quad a_1 = -1$

and $\quad a_n = \frac{a_{n-1}}{n}$, where $n \geq 2$

For $n = 2$, $\quad a_2 = \frac{a_{2-1}}{2} = \frac{a_1}{2} = \frac{-1}{2}$

For $n = 3$, $\quad a_3 = \frac{a_{3-1}}{3} = \frac{a_2}{3} = \frac{-1/2}{3} = -\frac{1}{6}$

For $n = 4$, $\quad a_4 = \frac{a_{4-1}}{4} = \frac{a_3}{4} = \frac{-1/6}{4} = -\frac{1}{24}$

For $n = 5$, $\quad a_5 = \frac{a_{5-1}}{5} = \frac{a_4}{5} = \frac{-1/24}{5} = -\frac{1}{120}$

So, first five terms are $= -1, -\frac{1}{2}, -\frac{1}{6}, -\frac{1}{24}, -\frac{1}{120}$ and sequence $= (-1) + \left(-\frac{1}{2}\right) + \left(-\frac{1}{6}\right) + \left(-\frac{1}{24}\right) + \left(-\frac{1}{120}\right) + ...$ **Ans.**

13. $a_1 = a_2 = 2, a_n = a_{n-1} - 1$, **where** $n > 2$

Sol. $\because \quad a_1 = a_2 = 2$

and $\quad a_n = a_{n-1} - 1$, where $n > 2$

For $n = 3$, $\quad a_3 = a_{3-1} - 1 = a_2 - 1 = 2 - 1 = 1$

For $n = 4$, $\quad a_4 = a_{4-1} - 1 = a_3 - 1 = 1 - 1 = 0$

For $n = 5$, $\quad a_5 = a_{5-1} - 1 = a_4 - 1 = 0 - 1 = - 1$

So, first five terms are 2, 2, 1, 0, – 1 and sequence = 2 + 2 + 1 + 0 + (–1) + ... **Ans.**

14. The Fibonacci sequence is defined by : $1 = a_1 = a_2$ **and** $a_n = a_{n-1} + a_{n-2}, n > 2.$

Find $\frac{a_{n+1}}{a_n}$, **for** $n = 1, 2, 3, 4, 5.$

Sol. $\because \quad a_1 = a_2 = 1$

and $\quad a_n = a_{n-1} + a_{n-2}, n > 2$

For $n = 3$, $a_3 = a_2 + a_1 = 1 + 1 = 2$

For $n = 4$, $a_4 = a_3 + a_2 = 2 + 1 = 3$

For $n = 5$, $a_5 = a_4 + a_3 = 3 + 2 = 5$

For $n = 6$, $a_6 = a_5 + a_4 = 5 + 3 = 8$

Now to get $\frac{a_{n+1}}{a_n}$,

For $n = 1$, $\quad \frac{a_{1+1}}{a_1} = \frac{a_2}{a_1} = \frac{1}{1} = 1$

For $n = 2$, $\quad \frac{a_{2+1}}{a_2} = \frac{a_3}{a_2} = \frac{2}{1} = 2$

For $n = 3$, $\quad \frac{a_{3+1}}{a_3} = \frac{a_4}{a_3} = \frac{3}{2}$

For $n = 4$, $\quad \frac{a_{4+1}}{a_4} = \frac{a_5}{a_4} = \frac{5}{3}$

For $n = 5$, $\quad \frac{a_{5+1}}{a_5} = \frac{a_6}{a_5} = \frac{8}{5}$

8.6 Geometric Progression

If each term of a series of non-zero number is obtained by multiplying by previous term by a fixed number, then series is called a geometric series *i.e.* the ratio of each term of a series to its previous term is constant, then the series is called geometric progression and the constant ratio is called the common ratio.

Examples : (i) 1, 4, 4^2, 4^3, ... is a geometric progression, whose common ratio is 4.

(ii) $\frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, -\frac{1}{81}$....... is a geometric progression whose common ratio is $-\frac{1}{3}$.

8.7 General term of a Geometric Progression

If first term of a progression is 'a' and common ratio is 'r', then

$a_1 = a$, $\quad a_2$ = first term × common ratio

$\Rightarrow \quad a_2 = ar$

a_3 = second term × common ratio $= ar \times r = ar^2$

a_4 = third term × common ratio $= ar^2 \times r = ar^3$

So, geometric progression is

$a_1 + a_2 + a_3 + a_4 + ...$

$\Rightarrow \quad a + ar + ar^2 + ar^3 + ...$

The series $a + ar + ar^2 + ... + ar^{n-1}$ or $a + ar + ar^2 + ... + ar^{n-1} + ...$ are called finite or infinite respectively.

The series $a + ar + ar^2 + ... + ar^{n-1}$ or $a + ar + ar^2 + ... + ar^{n-1} + ...$ are called finite or infinite geometric series, respectively.

8.8 n^{th} term of a Geometric Progression

First term $(a_1) = a = ar^{1-1}$

Second term $(a_2) = ar = ar^{2-1}$

Third term $(a_3) = ar^2 = ar^{3-1}$

Fourth term $(a_4) = ar^3 = ar^{4-1}$

....

....

....

....

n^{th} term $a_n = ar^{n-1}$

We represent the terms by T_1, T_2, T_3, T_4, ... T_n also.

Then $\boxed{T_n = ar^{n-1}}$

If last term is represented by l,

then last term $\boxed{l = ar^{n-1}}$

where $r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = ...$

8.9 Sum of the first n terms of a G.P.

Let the first term of a G.P. be a and the common ratio be r. Then the sum of first n terms of G.P.,

$$S_n = a + ar + ar^2 + ar^3 + ... + ar^{n-1} \quad ....(1)$$

Multiplying by r in equation (1)

$$rS_n = ar + ar^2 + ar^3 + ... + ar^{n-1} + ar^n \quad ....(2)$$

Subtracting equation (2) from equation (1)

$\Rightarrow \quad S_n - rS_n = a - ar^n$

$\Rightarrow \quad (1 - r)\, S_n = a\,(1 - r^n)$

$$\Rightarrow \quad S_n = \frac{a(1-r^n)}{1-r},\ r < 1$$

(i) If $r = 1$ then

$S_n = a + a + a + a + ...$ n terms

$S_n = na$

(ii) If $r > 1$ then

$$S_n = \frac{a(r^n-1)}{r-1}$$

(iii) If last term of geometric series is l, then

$l = ar^{n-1}$

$\Rightarrow \quad lr = ar^{n-1} \times r = ar^n$

From case (ii)

$$S_n = \frac{a(r^n-1)}{r-1} = \frac{ar^n-a}{r-1} = \frac{lr-a}{r-1}$$

$$S_n = \frac{lr-a}{r-1},\ r > 1$$

$$\Rightarrow \quad S_n = \frac{a-lr}{1-r},\ r < 1$$

Note : If $a + ar + ar^2 + ...$ is an infinite geometric progression,

(i) when $r > 1$ then $S_\infty = \infty$