According to the Latest NCERT Syllabus

Sanjiv®

Mathematics

Class-12 (Part-II)

For the Students of Rajasthan Board of Secondary Education

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❖ *Your suggestions shall be taken care of in our next edition.*

Preface

We are extremely pleased to present this book according to latest syllabus of NCERT. The book has been written in easy and simple language so that students may assimilate the subject easily. We hope that students will get benefitted from it and teachers will appreciate our efforts. In comparison to other books available in market, this book has many such features which make it a unique book :

1. Theoretical subject-material is given in adequate and accurate description along with pictures.
2. The latest syllabus of NCERT is followed thoroughly.
3. Complete solutions of all the questions given at the end of the chapter in the textbook are given in easy language.
4. Topic wise summary is also given in each chapter for the revision of the chapter.
5. In every chapter, all types of questions that can be asked in the exam (Objective, Fill in the blanks, Very short, Short, Numerical and Long answer type questions) are given.
6. At the end of every chapter, multiple choice questions asked in various competitive exams are also given with solutions.

Valuable suggestions received from subject experts, teachers and students have also been given appropriate place in the book.

We wholeheartedly bow to the Almighty God, whose continuous inspiration and blessings have made the writing of this book possible.

We express our heartfelt gratitude to the publisher – Mr. Pradeep Mittal and Manoj Mittal of Sanjiv Prakashan, all their staff, laser type center and printer for publishing this book in an attractive format on time and making it reach the hands of the students.

Although utmost care has been taken in publishing the book, human errors are still possible, hence, valuable suggestions are always welcome to make the book more useful.

In anticipation of cooperation!

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INTEGRALS

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7.1 Introduction

❖ If we study the history of integral calculus, we will know that it was discovered to find the areas of plane regions bounded by lines and curves in the plane. Its concept was based on the limit of sum of a series in which the number of terms was infinite and each term tended to zero. We will study about it from this prospective in Chapter 8. Later it was seen that, integration is the inverse process of differentiation.

❖ In differential mathematics, we have to find the differential cofficient or derivative of a given function, whereas in integral mathematics, we have to find the function whose differential coefficient is given. Therefore, the process by which we have to find the function F(x) from a given function $f(x)$ whose differential coefficient $\frac{d}{dx}\{F(x)\}$ = Function $f(x)$, is called integration. Obviously, this process of integration is the inverse process of differentiation and it is also called anti-derivatives. *i.e.* if

$$\frac{d}{dx}\{F(x)\} = f(x)$$

then F(x) is called the integration of functoin $f(x)$ with respect to x. Symbolically, it is represented as : $\int f(x)\,dx = F(x)$, where the symbol $\int$ (which is an extended form of the English alphabet S, which is used for addition) is used for integration and the letter x in dx expresses the variable with respect to which we have to integrate. The function $f(x)$ which we have to integrate is called integrand and is called the integral of F(x).

So, $\int f(x)dx = F(x)$

Integrand → Integral

Integration

☞ **For example :**

(i) Since $\frac{d}{dx}(\sin x) = \cos x \quad \therefore \int \cos x dx = \sin x$

(ii) Since $\frac{d}{dx}(\tan x) = \sec^2 x \quad \therefore \int \sec^2 x dx = \tan x$

(iii) $\frac{d}{dx}\left(\frac{x^5}{5}\right) = x^4 \quad \therefore \int x^4\, dx = \frac{x^5}{5}$

(iv) $\frac{d}{dx}\left(\frac{e^{3x}}{3}\right) = e^{3x} \quad \therefore \int e^{3x}\, dx = \frac{e^{3x}}{3}$

7.2 The Constant of Integration

- We know that the differentiation of any constant term is zero, *i.e.*

$$\frac{d}{dx}(c) = 0, \text{ where } c \text{ is a constant.}$$

Now if $\frac{d}{dx}\{F(x)\} = f(x)$

then $\frac{d}{dx}\{F(x) + c\} = \frac{d}{dx}\{F(x)\} + \frac{d}{dx}(c)$
$= f(x) + 0$
$= f(x)$

So by definition of integration,

$$\int f(x)\, dx = F(x) + c.$$

- There c is called the constant of integration and is independent from variable x. On putting different values of c, different integrals of the given function $f(x)$ are obtained, in which only constant term differs. So, $F(x) + c$ is called general integration of function $f(x)$.

- In general integration c is not constant. Due to uncertainty of c, it is also called indefinite integral.

☞ **For example :**

(i) Since $\frac{d}{dx}(\tan x + c) = \sec^2 x$

$$\therefore \int \sec^2 x\, dx = \tan x + c$$

(ii) Since $\frac{d}{dx}(e^x + c) = e^x$

$$\therefore \int e^x dx = e^x + c$$

(iii) Since $\frac{d}{dx}(\sin x + c) = \cos x$

$$\therefore \int \cos x\, dx = \sin x + c$$

- **Note :** Students should practice to write the constant of integration at the end of each indefinite integral. It is added at the end after the process of integration. For convenience, we do not write it again and again.

7.3 Theorems on Integration

- **Theorem 1 :** If constant of integration is removed then

$$\frac{d}{dx}\left[\int f(x)dx\right] = f(x).$$

Proof : Let

$$\frac{d}{dx}\{F(x)\} = f(x) \quad ...(1)$$

then according to definition of integration,

$$\int f(x)dx = \{F(x) + c\} \quad ...(2)$$

where c is a constant.

On differentiating eq. (2) with respect to x, both sides

$$\frac{d}{dx}\left[\int f(x)dx\right] = \frac{d}{dx}\{F(x)+c\}$$

$$= \frac{d}{dx}\{F(x)\}+\frac{d}{dx}(c)$$

$$= f(x) + 0 \quad \text{[From eq. (1)]}$$

$$\therefore \frac{d}{dx}\left[\int f(x)dx\right] = f(x)$$

Therefore this theorem proves that the process of integration is the inverse process of differentiation.

- **Theorem 2 :** The integration of product of a constant and a function is equal to the product of constant and integral of the function.

i.e. $\int kf(x)dx = k\int f(x)dx$

where k is a constant.

Proof : From the theorem of differential mathematics, we know that :

$$\frac{d}{dx}\left[k\int f(x)dx\right] = k\frac{d}{dx}\left[\int f(x)dx\right]$$

$$= kf(x) \quad \text{[From theorem 1]}$$

Thus according to definition of integration,

$$\int kf(x)dx = k\int f(x)dx.$$

- **Theorem 3 :**

$$\int\{f_1(x) \pm f_2(x)\}dx = \int f_1(x)dx \pm \int f_2(x)dx$$

i.e. integral of sum or difference of two functions is equal to sum or difference of their integrals.

Proof :

Let $\int f_1(x)dx = \phi_1(x)$ ...(1)

and $\int f_2(x)dx = \phi_2(x)$ ...(2)

so $\frac{d}{dx}\phi_1(x) = f_1(x)$ and $\frac{d}{dx}\phi_2(x) = f_2(x)$

$$\therefore \frac{d}{dx}\{\phi_1(x) \pm \phi_2(x)\} = \frac{d}{dx}\phi_1(x) \pm \frac{d}{dx}\phi_2(x)$$

$$= f_1(x) \pm f_2(x)$$

Integrating,

$$\phi_1(x) \pm \phi_2(x) = \int \{f_1(x) \pm f_2(x)\}dx$$

or $\int f_1(x)dx \pm \int f_2(x)dx = \int \{f_1(x) \pm f_2(x)\}dx$

[From eqns (1) and (2)]

This theorem can also be used to find the integration of finite number of terms.

$\therefore \int\{f_1(x) \pm f_2(x) \pm f_3(x)\} \pm ... \pm f_n(x)\}dx$

$= \int f_1(x)dx \pm \int f_2(x)dx \pm \int f_3(x)dx \pm ... \pm \int f_n(x)dx$

Thus the integration operation is distributed over a finite number of terms.

❖ **Comment :**

(1) This theorem is called the law of term-by-term integration.

(2) This rule does not necessarily apply to the integral of the sum of infinite terms.

❖ **General form :**

$\int a\, f_1(x)dx \pm \int b\, f_2(x)dx = a \int f_1(x)dx \pm b \int f_2(x)dx$

7.4 Standard Formulae of Integration

❖ By simply using the definitions in the standard formulas of differential mathematics, the corresponding formulas for integration can be obtained. This is shown in the following table :

	Differentiation Formula	**Corresponding Integration Formula**
1.	$\frac{d}{dx}(c) = 0$, c is a constant	$\int 0\, dx = c$, constant
2.	$\frac{d}{dx}\left(x^n\right) = n\, x^{n-1}, n \neq 0$ or $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n,\ n \neq -1$	$\int nx^{n-1}dx = x^n + c, n \neq 0$ or $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$
3.	$\frac{d}{dx}\left(\log\lvert x\rvert\right) = \frac{1}{x}, x \neq 0$	$\int \frac{1}{x} dx = \log\lvert x\rvert + c,\ x \neq 0$
4.	$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + c$
5.	$\frac{d}{dx}\left(\frac{a^x}{\log_e a}\right) = a^x,\ a > 0, a \neq 1$	$\int a^x dx = \frac{a^x}{\log_e a} + c\ =\ a^x \log_a e + c,\ a > 0,\ a \neq 1$
6.	$\frac{d}{dx}(\sin x) = \cos x$ where x is in radian	$\int \cos x\, dx = \sin x + c$ where x is in radian
7.	$\frac{d}{dx}(-\cos x) = \sin x$	$\int \sin x\, dx = -\cos x + c$
8.	$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\, dx = \tan x + c$
9.	$\frac{d}{dx}(-\cot x) = \text{cosec}^2 x = \csc^2 x$	$\int \text{cosec}^2 x\, dx = \int \csc^2 x\, dx = -\cot x + c$
10.	$\frac{d}{dx}(\sec x) = \sec x\, \tan x$	$\int \sec x \tan x\, dx = \sec x + c$
11.	$\frac{d}{dx}(-\text{cosec}\ x) = \text{cosec}\ x \cot x$	$\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + c$
12.	$\frac{d}{dx}(\sin^{-1} x) = \frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}},\ (\lvert x\rvert < 1)$	$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c = \arcsin x + c,\ (\lvert x\rvert < 1)$

13.	$\frac{d}{dx}(-\cos^{-1}x) = \frac{d}{dx}(-\arccos x) = \frac{1}{\sqrt{1-x^2}}$	$\int\frac{1}{\sqrt{1-x^2}}dx = -\cos^{-1}x + c$
14.	$\frac{d}{dx}(\tan^{-1}x) = \frac{d}{dx}(-\arctan x) = \frac{1}{1+x^2}$	$\int\frac{1}{1+x^2}dx = \tan^{-1}x + c = \arctan x + c$
15.	$\frac{d}{dx}\left(-\cot^{-1}x\right) = \frac{1}{1+x^2}$	$\int\frac{dx}{1+x^2} = -\cot^{-1}x + C$
16.	$\frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{x\sqrt{x^2-1}}$	$\int\frac{dx}{x\sqrt{x^2-1}} = -\sec^{-1}x + C$
17.	$\frac{d}{dx}\left(-\cos ec^{-1}x\right) = \frac{1}{x\sqrt{x^2-1}}$	$\int\frac{dx}{x\sqrt{x^2-1}} = -\text{cosec}^{-1}x + C$
18.	$\frac{d}{dx}\lvert x\rvert = \frac{\lvert x\rvert}{x},\ x \neq 0$	$\int\frac{\lvert x\rvert}{x}dx = \lvert x\rvert + c,\ (x \neq 0)$

- **Comment :** Integration constant c should be added in all formulas given above.
- **Abbreviated form :** cosec x = csc x, arc sin = $\sin^{-1}$ etc.

SOLVED EXAMPLES

☞ **Example 1.** Write an antiderivative for each of the following functions using the method of inspection :
(i) $\cos 2x$
(ii) $3x^2 + 4x^3$ **(Board Model Paper, 2022-23)**
(iii) $\frac{1}{x},\ x \neq 0$ **(NCERT)**

Sol. (i) Here we want to know the function whose derivative is $\cos 2x$.

We know that

$$\frac{d}{dx}(\sin 2x) = 2\cos 2x$$

$$\Rightarrow \quad \cos 2x = \frac{1}{2}\frac{d}{dx}(\sin 2x) = \frac{d}{dx}\left(\frac{1}{2}\sin 2x\right)$$

So antiderivative of $\cos 2x$ is $\frac{1}{2}\sin 2x$.

(ii) Here we want to know the function whose derivative is $3x^2 + 4x^3$.

Now, $\frac{d}{dx}\left(x^3 + x^4\right) = 3x^2 + 4x^3$

So antiderivative of $3x^2 + 4x^3$ is $(x^3 + x^4)$.

(iii) Here we want to know the function whose derivative is $\frac{1}{x}$.

So $\frac{d}{dx}(\log x) = \frac{1}{x},\ x > 0$

and $\frac{d}{dx}[\log(-x)] = \frac{1}{-x}(-1) = \frac{1}{x},\ x > 0$

On taking both simultaneously, we get

$$\frac{d}{dx}(\log \lvert x\rvert) = \frac{1}{x},\ x \neq 0$$

So $\int\frac{1}{x}dx = \log\lvert x\rvert$,

which is one of the antiderivative of $\frac{1}{x}$.

☞ **Example 2.** Find the integration of the following :

$$\int\frac{2\cos x}{3\sin^2 x}dx$$

Sol. Here, $\int\frac{2\cos x}{3\sin^2 x}dx = \int\frac{2\cos x}{3\sin x\cdot\sin x}dx$